Flow Past an Anchored Slender Ship in Variable-Depth Shallow Water

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The potential flow of a uniform stream of shallow water past a slender ship in the presence of a variable bottom is analyzed using the method of matched asymptotic expansions. For a small deviation from constant depth, the solution is seen to consist of the constant-depth result obtained by Tuck plus a correction which accounts for the depth variation. Expressions are obtained for the vertical force and trim moment to first order in the slenderness parameter when the flow is subcritical and the bottom is slender. It is seen that the bottom geometry appears in the solution in an analogous fashion to the hull geometry, through the streamwise variation of the crossflow cross section.

Introduction

THE problem under consideration is the potential flow of a uniform stream of shallow water past a slender ship in the presence of a variable bottom. Tuck[1] introduced the use of the method of matched asymptotic expansions into the solution of shallow-water hydrodynamics problems in his study of the longitudinal flow past a slender ship. Slender-body theory is used to describe the flowfield in the inner region near the ship and shallow-water theory is used in the flow description of the outer region. Tuck[2] also considered the case of a ship moving along the center of a constant width rectangular channel. Newman[3] solved the problem of the lateral flow past a slender ship in shallow water. Tuck[4,5] studied unsteady shallow-water problems related to ship motions.

In all of the previously mentioned research, the water depth is taken to be constant. It is intended here to extend the solution of Tuck[1] to include the effects of variable water depth. It is noted that the translation of a ship over a variable bottom is inherently unsteady, even in a ship-fixed coordinate system. As a first step in the understanding of the influence of depth variation on the forces and moments acting on a slender ship, the flow of a stream of shallow water past a fixed slender ship and a fixed variable-depth bottom is studied. This corresponds to the "water tunnel" version of the physical problem of interest.

Problem Formulation

Consider the steady subcritical potential flow of a stream of shallow water of speed U past a slender ship of length $2L$. The coordinate system is shown in Fig. 1. x lies in the stream direction and z is measured upwards from the undisturbed free surface. The slenderness assumption requires the beam and draft to be small compared to the length, say of $0(\epsilon)$. The hull is described by the equation

$$y=\epsilon f(x,z) \tag{1}$$

The shallowness assumption requires the depth to be $0(\epsilon)$ also and the bottom is given by

$$z=-\epsilon h(x,y) \tag{2}$$

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The depth Froude number

$$F_h^2=U^2/g\epsilon h$$

is taken to be of 0(1).

A perturbation velocity potential ϕ is defined which satisfies Laplace's equation and tends to zero at infinity. The total fluid velocity is

$$\bar{q}=U\nabla(x+\phi) \tag{3}$$

On the hull and bottom, the normal velocity component vanishes which yields

$$\phi_y-\epsilon(1+\phi_x)f_x-\epsilon\phi_z f_z=0 \quad \text{on } y=\epsilon f \tag{4}$$

and

$$\phi_z+\epsilon(1+\phi_x)h_x+\epsilon\phi_y h_y=0 \quad \text{on } z=-\epsilon h \tag{5}$$

Let the free surface be described by $z=\eta(x,y)$. The pressure is zero on the free surface and using the Bernoulli equation, this condition implies that

$$-2g\eta/U^2=2\phi_x+\phi_x^2+\phi_y^2+\phi_z^2 \tag{6}$$

The kinematic free-surface condition is

$$\phi_z=\eta_x+\phi_x\eta_x+\phi_y\eta_y \tag{7}$$

The method of matched asymptotic expansions will be used to solve the problem and the solution technique will closely parallel that of Tuck.[1]

Outer Expansion

In the outer region far from the ship, the coordinates have the following orders of magnitude

$$x,y=0(1), \quad z=0(\epsilon)$$

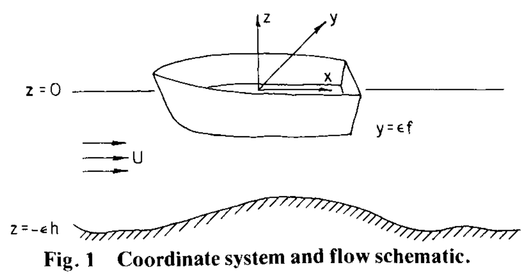

Fig. 1 Coordinate system and flow schematic.

The linearized shallow-water equation governing the motion is given in Wehausen and Laitone[6] as

$$(1-F_h^2)\phi_{xx}+\phi_{yy}+(1+\phi_x)h_x/h+\phi_y h_y/h=0 \qquad (8)$$

It is noted that this equation is valid for ϕ of $0(\epsilon)$ and that in the analysis to follow, terms of $0(\epsilon)$ and $0(\epsilon\delta)$ are kept. δ, a small parameter associated with the bottom variation, is defined below in Eq. (9). The terms of $0(\epsilon\delta)$ yield the "second-order" contribution to the flow due to the bottom variation and can consistently be obtained from Eq. (8). Terms of $0(\epsilon^2)$, which are independent of the bottom variation but which may be comparable to the $0(\epsilon\delta)$ terms, are obtained formally by Tuck[1] and can be added to the results of this analysis to yield a complete "second-order" solution.

The bottom is taken to deviate from constant depth by $0(\delta)$

$$h(x,y)=h_o+\delta h_1(x,y) \qquad (9)$$

Let

$$\phi(x,y)=\phi_o(x,y)+\delta\phi_1(x,y) \qquad (10)$$

Equations (9) and (10) are substituted into Eq. (8) and if terms linear in δ are kept, the equations for ϕ_o and ϕ_1 are

$$(1-F_o^2)\phi_{o_{xx}}+\phi_{o_{yy}}=0$$

$$(1-F_o^2)\phi_{1_{xx}}+\phi_{1_{yy}}=-h_{1_x}/h_0-[h_1(\phi_{o_{xx}}+\phi_{o_{yy}})+\phi_{o_x}h_{1_x}+\phi_{o_y}h_{1_y}]/h_o \qquad (11)$$

where F_o is the Froude number based on the mean depth ϵh_o. It is seen that ϕ_o satisfies the constant-depth shallow-water equation used by Tuck.[1]

Following Tuck,[1] the solutions are written formally in terms of Green's function distributions as

$$\phi_o=\int_{-\infty}^{\infty}dx_o\mu_1(x_o)G(x-x_o,y)$$

$$\phi_1=\int_{-\infty}^{\infty}dx_o\mu_2(x_o)G(x-x_o,y)-\int_{-\infty}^{\infty}dx_0\int_{-\infty}^{\infty}dy_o\{h_{1_x}+[h_1(\phi_{o_{xx}}+\phi_{o_{yy}})+\phi_{o_x}h_{1_x}+\phi_{o_y}h_{1_y}]\}\,G(x-x_o,y-y_o)/h_0 \qquad (12)$$

where

$$G(x,y)=\frac{1}{2\pi(1-F_o^2)^{1/2}}\log[x^2+(1-F_o^2)y^2]^{1/2} \qquad (13)$$

for subcritical flow. The unknown source strengths μ_1 and μ_2 are to be determined by matching with the inner expansion.

At this point, a slender bottom is assumed. The extent of the bottom variation is taken to be of $0(\epsilon)$ in the y-direction. The effect of this restriction on the double integral term in the expansion for ϕ_1 will now be demonstrated. First, since ϕ_o is of $0(\epsilon)$, terms in the integral proportional to ϕ_o may be neglected compared to h_{1_x}. Second, introduce the stretched variable $Y_o=y_o/\epsilon$ in the y_o integral.

$$\int_{-\infty}^{\infty}dy_oh_{1_x}G(x-x_0,\ y-y_o)=\epsilon\int_{-\infty}^{\infty}dY_oh_{1_x}G(x-x_0,\ y-\epsilon Y_o)$$

In the outer region $y=0(1)$ and a Taylor series of expansion of G yields

$$G(x-x_0,\ y-\epsilon Y_o)=G(x-x_o,y)+0(\epsilon)$$

If terms up to $0(\epsilon)$ in ϕ_1 are kept, for a slender bottom Eq. (12) becomes

$$\phi_1=\int_{-\infty}^{\infty}dx_oG(x-x_o,y)[\mu_2(x_o)-\int_{-\infty}^{\infty}dy_oh_{1_x}/h_0] \qquad (14)$$

Inner Expansion

In the inner region, the coordinates have the following orders of magnitude

$$x=0(1),\ y,z=0(\epsilon)$$

Inner variables Y and Z are defined as follows

$$Y=y/\epsilon,\quad Z=z/\epsilon \qquad (15)$$

Following the analysis of Tuck,[1] terms in ϕ to $0(\epsilon^2)$ will be kept. Conventional slender-body theory yields the following differential equation, hull boundary condition and free-surface condition:

$$\phi_{YY}+\phi_{ZZ}=0 \qquad (16)$$

$$\phi_N=\epsilon^2 f_x(1+f_Z^2)^{-1/2}\quad \text{on } Y=f \qquad (17)$$

$$\phi_Z=0\quad \text{on } Z=0 \qquad (18)$$

where N is the normal in inner variables in the $Y-Z$ plane. On the bottom, from Eq. (5),

$$\phi_Z+\phi_Yh_Y+\epsilon^2h_x=0\quad \text{on } Z=-h$$

Using Eq. (9) and a Taylor series expansion about $Z=-h_o$ yields

$$\phi_Z+\delta(h_1\phi_Y)_Y+\delta\epsilon^2h_{1_x}=0\quad \text{on } Z=-h_o \qquad (19)$$

For

$$\phi=\Phi_o+\delta\Phi_1 \qquad (20)$$

the following mathematical problems are obtained

$$\Phi_{O_{YY}}+\Phi_{O_{ZZ}}=0$$

$$\Phi_{O_Z}=0\quad \text{on } Z=0$$

$$\Phi_{O_N}=\epsilon^2 f_x(1+f_Z^2)^{-1/2}\quad \text{on } Y=f$$

$$\Phi_{O_Z}=0\quad \text{on } Z=-h_o \qquad (21)$$

$$\Phi_{1_{YY}}+\Phi_{1_{ZZ}}=0$$

$$\Phi_{1_Z}=0\quad \text{on } Z=0$$

$$\Phi_{1_N}=0\quad \text{on } Y=f$$

$$\Phi_{1_Z}=-(h_1\Phi_{o_Y})_Y-\epsilon^2h_{1_x}\quad \text{on } Z=0 \qquad (22)$$

Equations (21) and (22) represent a series of two-dimensional von Neumann problems to be solved at each cross section location x. Note that to $0(\epsilon)$, both Φ_o and Φ_1 admit arbitrary functions of x as solutions, and that the problem for Φ_o is the Tuck[1] constant-depth one. Let

$$\Phi_o=\epsilon g(x)+\Phi_{oo}(Y,Z;x)$$

$$\Phi_1=\epsilon r(x)+\Phi_{1o}(Y,Z;x) \qquad (23)$$

where Φ_{oo} and Φ_{1o} are $0(\epsilon^2)$.

We can solve both Φ_{oo} and Φ_{1o} uniquely if a suitable boundary condition at infinity is specified. If it is assumed that the hull and bottom are symmetrical with respect to the plane $Y=0$, appropriate conditions are $\Phi_{oo} \to u_1 |Y| + o(1)$ as $|Y| \to \infty$ and $\Phi_{1o} \to u_2 |Y| + o(1)$ as $|Y| \to \infty$ where u_1 and u_2 are determined by conservation of mass. Since the hull area flux is $\epsilon^2 S'(x)$ and the flux from the bottom is

$$-\epsilon^2 \delta \frac{d}{dx} \int_{-\infty}^{\infty} h_1 dY \equiv \epsilon^2 \delta A'(x) \tag{24}$$

then

$$u_1 = \epsilon^2 S'(x)/2h_0 \,,\quad u_2 = \epsilon^2 \delta A'(x)/2h_o \tag{25}$$

where $\epsilon^2 S(x)$ is the hull cross section beneath $Z=0$ and $\epsilon^2 \delta A(x)$ is the net bottom cross section which is positive for h_1 negative.

Matching

To determine the unknown functions μ_1, μ_2, g and r, the inner and outer expansions must be matched. The following matching principle from Van Dyke[7] is used:

> "The m-term inner expansion of the (n-term outer expansion) = the n-term outer expansion of the (m-term inner expansion)" (26)

Take $m=2$ and $n=1$. The 1-term outer expansion is $\phi_o + \delta\phi_1$. It has a 2-term inner expansion of

$$\begin{aligned}\phi_o(x,o) + \delta\phi_1(x,o) + |y|[\phi_{o_y}(x,o+) + \delta\phi_{1_y}(x,o+)] \\ = \phi_o(x,o) + \delta\phi_1(x,o) + |y|[\mu_1(x) \\ + \delta\mu_2(x) + \epsilon\delta A'(x)/h_o]/2\end{aligned} \tag{27}$$

The 2-term inner expansion is $\epsilon g(x) + \epsilon\delta r(x) + \Phi_{oo} + \delta\Phi_{1o}$. It has a 1-term outer expansion of

$$\epsilon g(x) + \epsilon\delta r(x) + \epsilon |y| [S'(x) + \delta A'(x)]/2h_o \tag{28}$$

By equating Eq. (28) to Eq. (27), the results of the matching are obtained as

$$\begin{aligned}\mu_1(x) = \epsilon S'(x)/h_o, \quad \mu_2(x) = 0 \\ \epsilon g(x) = \phi_o(x,o), \quad \epsilon r(x) = \phi_1(x,o)\end{aligned} \tag{29}$$

Inner Expansion of the Pressure and Forces

A hydrodynamic pressure coefficient is defined and from Bernoulli's equation

$$C_p = p/\tfrac{1}{2}\rho U^2 = -2\phi_x - \phi_x^2 - \phi_y^2 - \phi_z^2 \tag{30}$$

To $0(\epsilon)$, the inner velocity potential is given in Eqs. (23) as

$$\phi = \epsilon g(x) + \epsilon\delta r(x)$$

and therefore, to $0(\epsilon)$,

$$C_p = -2\epsilon g'(x) - 2\epsilon\delta r'(x)$$

To first order in ϵ, the pressure [Eqs. (12 and 29)] is

$$p = \frac{\rho U^2 \epsilon}{2\pi h_o (1-F_0^2)^{1/2}} \int_{-\infty}^{\infty} dx_o \left[\frac{S'(x_o) + \delta A'(x_o)}{x - x_o} \right] \tag{31}$$

As in Tuck,[1] to this order the pressure is a function of x only. The hull and bottom geometry appear only through the cross-sectional areas S and A. The constant-depth solution of Tuck[1] is recovered for $A=0$.

Tuck[1] shows that the vertical force (positive upwards) is

$$F = \int_{-L}^{L} dx\, p(x) B(x) \tag{32}$$

and the trim moment (positive clockwise) is

$$M = -\int_{-L}^{L} x\, dx\, p(x) B(x) \tag{33}$$

where $B(x)$ is the beam. Force and moment coefficients which are independent of F_o can be defined as

$$\begin{aligned} -F = 2\rho g L \int_{-L}^{L} B(x) dx\, F_o^2 (1-F_o^2)^{-1/2} C_F \\ M = \rho g \int_{-L}^{L} x^2 B(x) dx\, F_o^2 (1-F_o^2)^{-1/2} C_M \end{aligned} \tag{34}$$

Elliposidal Hull and Bump

For example consider an elliposidal hull of revolution described by $x^2/L^2 + Y^2 + Z^2/B_o^2 = 1$.

The area $S(x)$ and the beam $B(x)$ are then $S = 0.5\pi B_0^2 (1 - x^2/L^2)$ and $B = 2B_o \epsilon (1 - x^2/L^2)^{1/2}$. Using Eq. (31), the pressure coefficient is

$$C_p = \frac{\epsilon B_o}{h_o L (1-F_o^2)^{1/2}} \left[\frac{x}{L} \log \left| \frac{1 + x/L}{1 - x/L} \right| - 2 \right]$$

The pressure distribution is plotted in Fig. 2. From Eqs. (32) and (33),

$$F = -\frac{2\pi}{3} \frac{\rho U^2 \epsilon^2 B_o^3}{h_o (1-F_o^2)^{1/2}} \,,\quad M = 0$$

and from Eqs. (34) $C_F = B_o^2 \epsilon^2 / 3L^2$, $C_M = 0$. Note that the force is downward.

Consider an ellipsoidal bump with semi-axes given in inner variables as x_B, Y_B and δh_B and centered at $x = x_C$. The net cross-sectional area is then

$$A(x) = 0.5 h_B Y_B [1 - (x - x_c)^2 / x_B^2]$$

Using Eq. (31), the pressure coefficient is

$$C_p = \frac{\epsilon \delta h_B Y_B}{h_o x_B^2 (1-F_o^2)^{1/2}} \left[(x - x_c) \log \left| \frac{1 + \frac{x - x_c}{x_B}}{1 - \frac{x - x_c}{x_B}} \right| - 2x_B \right]$$

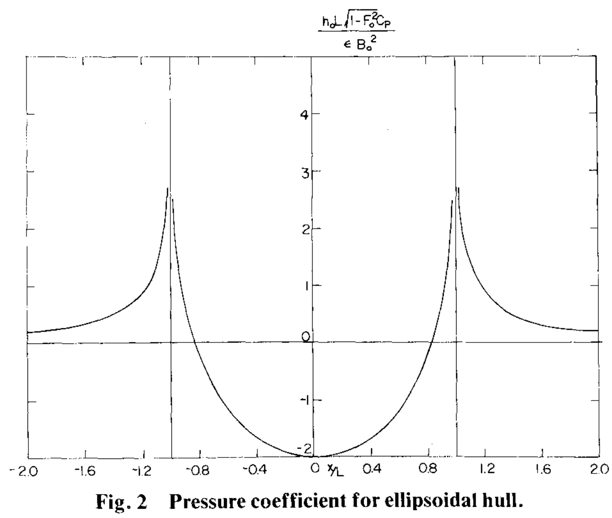

Fig. 2 Pressure coefficient for ellipsoidal hull.

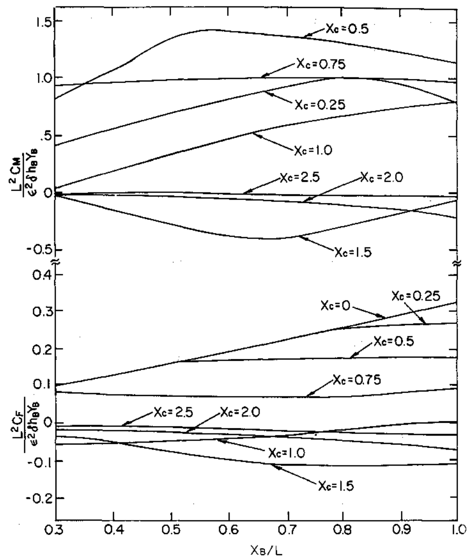

Fig. 3 Force and moment coefficients, C_F and C_M, for ellipsoidal bump as functions of X_B/L, the bump length, and X_C/L, the bump location.

C_F and C_M for the bump are computed using Eqs. (32-34) and are plotted in Fig. 3 as functions of x_B/L, the bump length, and x_c/L, the bump location. For $x_c/L = 0$, the force is downward and the moment is zero. As x_c/L increases, the force is seen to change sign and to eventually decay to zero. As x_c/L increases, a moment is generated which is initially positive and which then changes sign and decays to zero.

Conclusions

The solution of Tuck[1] for the potential flow past a slender ship in shallow water has been extended to include the effect of depth variation. For a slender bottom and symmetrical hull and bottom, the forces to first order in the slenderness parameter are obtained by quadrature. The results are presented as corrections to the constant-depth solution and the bottom geometry enters the solution in the same fashion as the hull geometry. Physically, the solution to this order depends only on the streamwise rate of change of the net cross-sectional wetted area. Solutions are presented for subcritical flow but can easily be obtained for supercritical flow. For bottom geometry asymmetrical with respect to the hull symmetry plane, it is necessary to solve the inner problem in detail.

Tuck has demonstrated that this solution technique yields physically reasonable results for the vertical force and trim moment for the constant-depth case. His solution corresponds to the physically interesting problem of a ship translating in shallow water. With depth variation included, the solution obtained here does not correspond to the ship translation problem which is unsteady. At best, by solving a series of steady problems with the bottom geometry shifted in the streamwise direction (as in the example), a quasi-steady approximation to the translation problem can be obtained.

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